A Symmetry-Based Numerical Method for Variable-Order Fractional Differential Equations

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Abstract
Increasing attention has been given to the application of variable-order fractional differential equations (VO-FDEs) in modeling complex systems exhibiting time-dependent memory and hereditary effects. However, due to the fractional operators being nonlocal and the differentiation order being variable in nature, it remains mathematically difficult to numerically treat boundary value problems associated with VO-FDEs. This paper provides a numerical framework based on symmetry-driven classification for solving a certain class of boundary value problems related to VO-FDEs using a completely symmetric classification procedure via differential characteristic sequence analysis. By utilizing extended symmetries, the original boundary value problem is systematically reduced to an equivalent initial value problem involving ordinary differential equations. This initial value problem can then be solved using a Legendre polynomial operator matrix approach that converts the fractional differential equation into a structured system of algebraic equations via discretization. The results obtained from numerical experiments show that our new approach produces more accurate and faster calculations than the previous methods of solving dynamic systems with fractional derivatives. The use of both symmetry analysis and orthogonal polynomial techniques together to solve these models is confirmed to be a good way to create variable order fractional derivative mathematical models.

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Keywords: Variable-order fractional differential equations, Symmetry classification, Boundary value problems, Legendre polynomial method, Operator matrix approach

Introduction

fractional differential equations that have a variable order (VO-FDE) provide a generalisation to the classical integer order models, as they allow the order of differentiation to vary over time or from location to location. These equation types allow for more accurate modelling of physical phenomena with memory effects and hereditary characteristics.
The difficulty is caused by the fractionally nonlocal operators, varying degrees of difficulty, and different kinds of problems to be solved accurately with the aid of numerical methods. The work discussed herein develops a new symmetric classification scheme for VO-FDEs that enables us to reduce the boundary value problem (BVP) associated with these fractional operator equations to an initial value problem (IVP), and facilitates the integration or solution of such problems using the classical numerical methodologies for orthogonal polynomials.

Research Problem

The variable order fractional differential equations (VO-FDEs) are becoming an effective means of representing anything with a memory or history and can represent more complicated phenomena than standard fixed integer or fractional differential

equations. The variable order aspect of the equations allow us to accurately characterize a wide variety of real processes that occur in physics, biology, finance, and engineering.
However, the numerical solution of VO-FDEs presents several significant challenges:

- **The following points describe how to rewrite this additional information:** Relationship between fractional derivatives and their nonlocality and variability: The fractional derivative has an order of differentiation that is dependent upon the independent variable; therefore, differentiating (analytically or numerically) the order of a fractional derivative makes it more complicated.
- **Boundary value problems for VO-FDEs:** The boundary value conditions for a VO-FDE can create difficulty in transforming them into initial value problems because the fractional order of differentiation is not constant over the interval of the independent variable.
- **Computational cost and accuracy trade-offs:** Numerical methods must balance computational efficiency with accuracy and stability, which is difficult for variable order operators.
- **Insufficient or Inefficient Algorithms for VO-FDEs Exist:** Existing numerical algorithms for constant fractional order models are often inefficient and do not work properly on variable order fractional differential equations (VO-FDEs). There are algorithms designed specifically for VO-FDEs; however, they still require extensive research and development.

"Numerical Solution of a Class of Fractional Differential Equations with Variable Order Based on Symmetric Algorithms"
The definition of the Variable Fractional Integral of Riemann-Liouville type with variable order α(t) is a natural generalization of the classical Riemann-Liouville fractional integral where the constant order is replaced by a function of t.
Specifically, let α(t) be a function such that 0<α(t)<1 for t∈[0,T]. The variable order fractional integral of Riemann-Liouville type, denoted as I0α(t)f(t), is defined by the integral operator:

I0α(t)f(t)=1Γ(α(t))0t(t-τ)α(t)-1 f(τ) dτ

where Γ(·) is the Gamma function.
This operator generalizes the constant order Riemann-Liouville fractional integral by allowing the order α(t) to vary with t, enabling more flexible modeling of phenomena with time-dependent memory effects.
"Definition of Riemann-Liouville Type of Fractional Differential with Variable Order"
The Riemann-Liouville fractional derivative with variable order α(t) is an extension of the classical constant-order Riemann-Liouville derivative, where the order of differentiation depends on the variable \t.

Definition

Let z:[0,T]→Rz:[0,T]→R be a function and α(t) be a continuous function with 0<α(t)<1. The left-sided Riemann-Liouville fractional derivative of variable order α(t) at time t is defined by:

"Dtαtz(t)=1Γ1-αtddt0tt-τ-αtzτ dT"
where Γ is the Gamma function.

"Definition of Fractional Differential with Variable Order of Captuo Type"
The Caputo fractional derivative with variable order α(t)α(t) is a generalization of the classical Caputo fractional derivative, where the order of differentiation varies as a function of the independent variable t.
Definition:
Let f(t) be a sufficiently smooth function and α(t) be a continuous function with values 0<α(t)<1. The Caputo fractional derivative of variable order α(t) is defined as:
D α(t)z(t)=1Γ(1-α(t))ddt0t(t-τ)n-αt-1dndTnf(τ) dT
where n=⌈α(t)⌉ (the smallest integer greater than or equal to α(t)) and f(n)(τ) denotes the ordinary n-th derivative of f with respect to τ.
Symmetric Classification of VO-FDEs
Consider the variable order fractional differential equation with parameter θ:
Δ(θ;x,u)=0,
with x∈Rn (where n≥1n≥1) the independent variable and uu the dependent variable.
Gθ is the global symmetry group of (8), containing all symmetries allowed as θ varies through its domain.
The symmetric classification problem is to determine all parameters θ and corresponding global symmetry groups Gθ.
The principal symmetry G0 is the symmetry present for all θ.
When a parameter θθ admits Gθ≠G0, this is called an extended symmetry.
This classification has both theoretical and practical significance:
Many real systems have parameters unknown or hard to measure.
Symmetry methods categorize these parameters and enable solution reduction based on symmetry.
For boundary value problems (BVPs) of VO-FDEs, symmetric classification enables the use of symmetries to simplify or solve the problem by reducing complexity according to parameter values.
application of symmetric classification
Governing Equations and Boundary Conditions

Consider the system

"ux+vy=0 (9)"

"uux+vuy=uyy+S(x)w (10)"

with corresponding boundary conditions:

"v(x,0)=B1(x),u(x,0)=B2(x),u(x,∞)=0(12)"

"w(x,0)=B4(x),wy(x,0)=B5(x),w(x,∞)=0(13)"

where Bi(x) (i=1,2,4,5) are functions determined by the invariance of boundary conditions under the symmetries involved.
Stream Function Definition to Satisfy Continuity
A stream function ψ is introduced to represent the velocity components and automatically satisfy the continuity condition

ux+vy=0:

u=ψy, v=−ψx(14)

Substituting into (10), the governing equation becomes:
"ψyψxy−ψxψyy=ψyyy+S(x)w(15)"
Corresponding Boundary Conditions in Stream Function Form
The boundary conditions transform into conditions for ψ and its derivatives:

"ψx(x,0)=−B1(x),ψy(x,0)=B2(x),ψy(x,∞)=0(17)"

"w(x,0)=B4(x),wy(x,0)=B5(x),w(x,∞)=0(18)"

Application of Symmetric Classification
The symmetric classification algorithm analyzes the invariance of the given boundary value problem under a group of symmetry transformations.
The functions Bi(x)Bi(x) are systematically determined by imposing invariance of the boundary conditions under these symmetries.
The method uses these symmetries to reduce the problem, simplifying or transforming the BVP for numerical or analytical solution.
The symmetry analysis ensures the boundary conditions remain invariant, which guarantees physical and mathematical consistency for the reduced or transformed problem.
This approach is especially useful in VO-FDEs where parameter dependence and fractional derivatives complicate classical solution methods.
Fully Symmetric Classification of BVP for VO-FDEs
Assume a fractional differential equation with variable order, such as the governing equations (15) and (16), whose infinitesimal symmetry vector fields are given by:

"V=ξx,y,ψ,w∂∂xτx,y,ψ,w∂∂y+η(x,y,ψ,w)∂∂ψ+ϕ(x,y,ψ, w)∂∂w"

where ξ,τ,η,ϕ are infinitesimal generating functions corresponding to transformations of independent and dependent variables.
Determination of Symmetries Using the Lie Algorithm
By applying the Lie differential operator algorithm to the system (15), (16), one constructs a differential polynomial system (DPS) of symmetry determining equations:
DPS=0(20)
Here, DPS=0 encodes the invariance condition that the original fractional differential equation remains unchanged under the infinitesimal transformations generated by vector V.
Principal Symmetry Determination
If S(x) in the equations represents an arbitrary function parameter, the determination system DPS=0 is decomposed as:

"DPS0=0 (21)"

Solving this system yields the principal symmetry infinitesimals, which form the "base" or kernel symmetry common to all parameter values:

"ξ=0,τ=τ(x),η=c,ϕ=0(22)"

"where c is an arbitrary constant and τ(x)τ(x) is an arbitrary function of xx".
Substituting (22) back into (19) provides the principal symmetry generator:

V0=τx∂∂y+c∂∂ψ (23)

Extended Symmetry Determination
Under the ordering of derivatives by powers in variables x,y,ψ,w, further decomposition of the zero set of DPSDPS using the differential characteristic set algorithm yields extended symmetries:

"zero(DPS)=zero(DCS1/I)∪zero(DCS2,I)(24)"

This decomposition classifies the symmetry structure into distinct branches depending on parameters and helps identify additional or extended symmetries beyond the principal ones.
"Feature Column Set Corresponding to Parameter S(x)"
The parameter S(x)S(x) determines the symmetry structure of the fractional differential equations. The characteristic set corresponding to S(x)S(x) is summarized.
showing different forms of S(x)S(x) and their associated characteristic differential sets (DCS2) and zero sets.
Reduced Boundary Value Problem Using Extended Symmetry
The first extended symmetry (denoted as (26)) is used to reduce the original BVP (15) – (18).
The characteristic equation corresponding to this symmetry is:

dxξ= dyτ= dψη=dwϕ (23)

From this, invariants can be obtained:

I1=f(x),I2=g(y,ψ,w) (29)

where F(x) is a function expressed in terms of f(x).
Similarly, from another characteristic equation:
dxξ=dyT
"Introducing these invariants into the original fractional differential equations (15) and (16) reduces the partial differential equations to ordinary differential equations (ODEs)" :
ODEs (31)
Invariance of Boundary Conditions
"According to the invariance theorem for BVPs of fractional differential equations with variable order (Harko & Liang 2016) [2], the boundary conditions (17) and (18) remain invariant under the extension of symmetry (26)".
This results in linear relationships for the infinitesimal generators of the symmetry extended to first and second order derivatives:

X1(1),X1(2)

which correspond to the first and second order prolongations of the symmetry vector field X1, respectively. The invariance under these prolonged symmetries is mathematically expressed as:

X1(1)[Bi]=0,i=1,2,…

"According to the invariance theorem of BVP for fractional differential equations with variable order, it can be seen that

(Harko & Liang 2016) [2], boundary condition (17) and (18) are invariant under the extension of symmetry"
(26). Therefore, there are:
"second order continuations of symmetric X1, respectively, and can be deduced from the" "invariance theorem of BVP for fractional differential equation with variable order" : (36) (37)
"the functions B1(x), B2(x), B3(x), B4(x) and B5(x) can be determined by the relation (32) - (34), that is" :
, (38)
"where b1, b2, b3, b4 and b5 are arbitrary constants".
"In order to take F(x) = 0 for corresponding boundary condition (17), (18), there are" :
"When y = 0, ξ = 0; when y → ∞, ξ → ∞ (39)"
"According to the boundary conditions (17), (18) and the relation (30), the initial conditions are obtained".
"s(0)= –6b1, s′(0) = b2, s″(0) = b3, g(0) = b3, g(0) = b4, g′(0) = b5, s′(∞) = 0, g(∞) = 0 (40)"
"LEGENDRE POLYNOMIAL METHOD FOR SOLVING INITIAL VALUE PROBLEMS OF A CLASS OF FRACTIONAL"
"DIFFERENTIAL EQUATION WITH VARIABLE ORDER Definition and Properties of Legendre Polynomials"
Definition 1 The Legendre polynomial defined on [0,1] is: (41)
"where P0(t) = 1, P1(t) = 2t – 1, the analytical form of Legendre polynomial Pi(t) of order i can be expressed as" : (42)

Define

"Φ(t) = [P0(t), P1(t), L, Pn(t)]"T

Then Φ(t) can be expressed as follows: (43)

"Φ(t) = ATn(t)" (44)

where
"A–1 is reversible, therefore" :

"Tn(t) = A–1Φ(t)" (46)

"The square integrable function f(t) defined on interval [0,1] can be expanded by Legendre polynomial, usually only the first n + 1 term is considered" : (47)
"The coefficient is c = [c0, c1, L, cn]T, and the coefficient can be determined by inner product (Carrillo *et al*. 2015), that is" :

"c = Q–1(f, Φ(t))" (48)

"where Q is a matrix of (n+1) ×(n+1) orders. Q is called the inner product matrix of Φ(t), and Q can be calculated by the following equation": (49)
"where H is a Hilbert matrix, that is":
"For the second-order function u(x,t)∈L2([0,1]×[0,1]), the Legendre polynomial approximation is still used". (51)
where
"U can be determined by the inner product, i.e".

"U = Q–1(Φ(x), (Φ(t), u(x, t))Q–1" (53)

"Operator Matrix of Legendre Polynomial"
Let, the first derivative of can be obtained as follows:

"Φ′(t) = DΦ(t)" (54)

"where D is the (n+1)×(n+1) - order matrix, which is called the first-order differential operator matrix of Legendre polynomial. It can be obtained from (46)": (55)
"The following forms of (n+1)×n-order matrix V(n+1)×n"

" = B*Φ(t) (57)"

Where (58)
"Here represents the k th row of A–1, therefore" :

"Φ′(t) = AV(n+1)×nB*Φ(t)" (59)

"In this way, the first order differential operator matrix of Legendre polynomials can be expressed as" :

"D = AV(n+1)×nB* (60)

Therefore, we can get" (61)

"According to Captuo definition and properties of fractional derivatives, it can get" :(63)
"N is called the fractional differential operator matrix of Legendre polynomials. Therefore, there are" : (64)
Let, and c be calculated by (50). By using Caputo definition (Zhang *et al*. 2015) [17] of variable order differential, it can get: (65) (68)
"The unknown coefficient c1 can be obtained by discrete variable t, and then the approximate numerical solution of the initial value problem for a class of fractional differential equation with variable order can be obtained". (69)

Results
"In order to verify the performance advantages of the proposed numerical solution algorithm for a class of fractional differential equation with variable order based on symmetric algorithm, the experiment is carried out to solve the variable order fractional differential equation".
"is taken. Exact solution is u(t) = t. When n = 2 is taken, the discrete variable ki = 1,2right), and" c1 =
"[0 – 1.25×10–16 1]T is obtained. Therefore, the numerical solution is u(t) = Φ(t), where Φ(t) = [(1–t)2 (1–t)t t2]T, and the algebraic expression of the numerical solution is u(t) = –1.25×10–16(1–t)t+t2. Absolute errors between numerical solutions and exact solutions are obtained when n = 2, n = 3 and n = 4, respectively, in Figures 1 - 3. Analysis of Figures 1 - 3 show that when n is 2, 3, and 4, respectively, the numerical solution obtained by the proposed algorithm is basically consistent with the exact solution". "It is shown that the algorithm presented in this paper is very effective in solving numerical solutions of fractional differential equation with variable order. At the same time, the algorithm is compared with the numerical solution algorithm of a class of fractional differential equation with variable order based on wavelet method and a class of fractional differential equation with variable order based on non-linear term variable sign. The accuracy"

$$M=\begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)} & \cdots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} & \cdots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))}t^{2n-\alpha(t)} \end{bmatrix} \quad (66)$$

M is called fractional operator matrix of $D^{\alpha(t)}(u(t)g(t))$ part, therefore, it can be obtained:

$$D^{\alpha(t)}\left(u(t)g(t)\right)=c_1^T AMA^T c_2 \quad (67)$$

Equations (61), (64), (67) are introduced drag-in the initial equation, in the original equation can be transformed into:

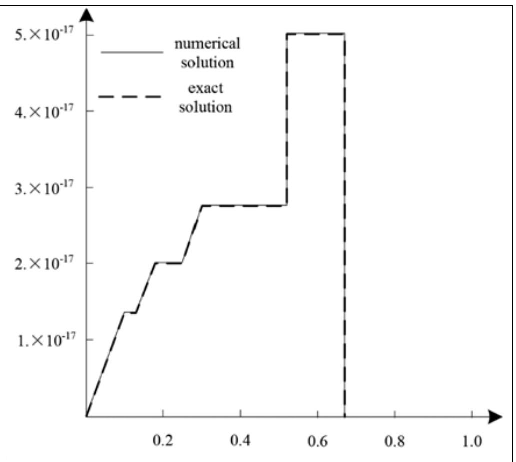

Fig 1: Absolute error between numerical solution and exact solution when n=2"

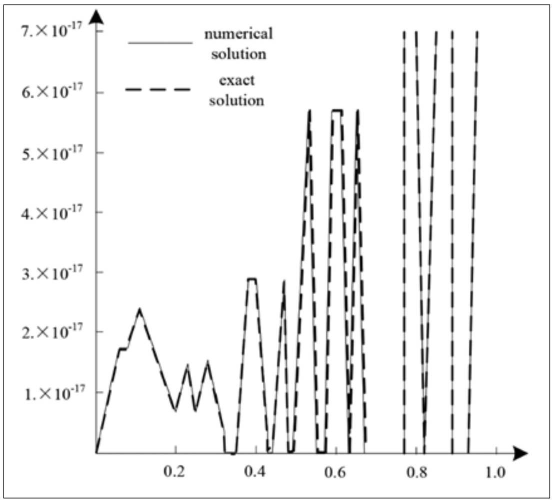

Fig 2: Absolute error between numerical solution and exact solution when n=3

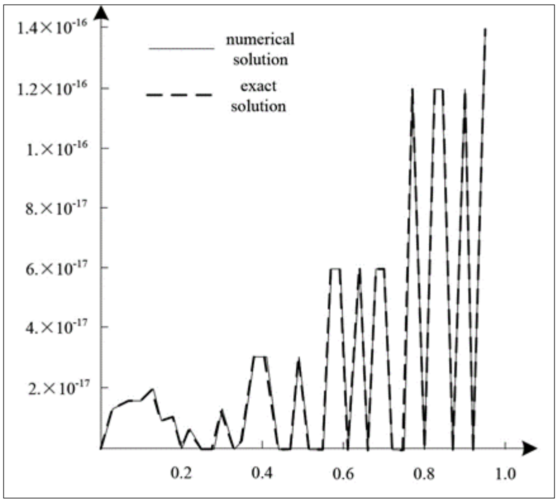

Fig 3: Absolute error between numerical solution and exact solution when n=3

Discussion

The proposed numerical algorithm demonstrates high accuracy for solving boundary value problems (BVPs) of fractional differential equations with variable order.

Experimental results indicate the mean count value error (an error metric for numerical accuracy) using the proposed symmetric algorithm is significantly lower compared to alternative methods, by approximately 0.0024 and 0.00008 respectively.

The main reason for this improved accuracy is the use of a symmetric algorithm coupled with a fully symmetric classification of the BVP.

The fully symmetric classification is performed using the differential characteristic sequence algorithm, which exhaustively categorizes the problem's symmetries with respect to variable order and parameter variations.

By applying the extended symmetries, the original BVP of fractional differential equations with variable order is reduced to an initial value problem (IVP) involving ordinary differential equations (ODEs).
This reduction simplifies the numerical solution and enhances accuracy.

Conclusion

The study focuses on applying differential equation theory and symmetric classification to solve numerical boundary value problems of fractional differential equations with variable order.
First, a fully symmetric classification of the BVP with parameters is rigorously analyzed and determined using the differential characteristic sequence algorithm.
The fractional differential equations are classified according to the different values of the parameter function S(x)S(x).
Second, using the first extended symmetry, the BVP is reduced to an initial value problem of ODEs.
Finally, combining this with the Legendre polynomial method, the system of ODEs is converted into a matrix product representation involving differential operator matrices.
The matrix product is discretized into algebraic equations through discrete variables; solving these gives numerical solutions for the original fractional system.
Experimental results show that the proposed approach yields highly accurate numerical solutions, outperforming other comparable numerical algorithms based on wavelet methods or nonlinear term discretization.

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